Parallel and Distributed Algorithms

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Assignment 7

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7.1. Problem (8) *The Metropolis algorithm*

Assume that the neighborhoods are symmetric

(i.e., $x \in U(x)$ for all $x \in \Omega, y \in N(x)$ whenever $x \in N(y)$ and $|U(x)| = |U(y)|$ for all $x, y \in \Omega$).

The assumption $|U(x)| = |U(y)|$ for all $x, y \in \Omega$ will simplify the argument. Moreover assume that there is a path from x to y for any pair $(x, y) \in \Omega^2$. We fix the temperature T. Then the Markov chain (Ω, P_T) has a unique stationary distribution π.

a.) Set $q_T = \frac{e^{\frac{-f(x))}{T}}}{N_T}$ with $N_T = \sum_{x\in\Omega} e^{\frac{-f(x))}{T}}$ and define $P_T[x, y]$ to be the probability to move from state x to state y. Show that q_T is reversible, i.e.,

$$q_T(x) \cdot P_T[x, y] = q_T(y) \cdot P_T[y, x]$$

holds.

Hint: Recall that if the Metropolis algorithm currently visits point $x \in \Omega$, it randomly picks a neighbor $y \in N(x)$. If $f(y) \leq f(x)$ holds, the algorithm continues with y with probability 1. In the case $f(y) > f(x)$ the point y is accepted with probability $e^{-\frac{f(y)-f(x)}{T}}$.

b.) Show that any reversible distribution ρ coincides with the stationary distribution. Thus q_T coincides with the stationary distribution π.

Hint: Consider the equation $\sum_j q_i \cdot P[i, j] = \sum_j q_j \cdot P[j, i]$.

7.2. Problem (8) *Backtracking applied on the Satisfiability problem*

We redefine our backtracking algorithm for the Satisfiability problem (cf. example 5.11 on pp. 102 of the lecture notes).

This time, to disqualify partial assignments we consider all clauses with at most two unassigned literals. If these clauses cannot be satisfied simultaneously, we disqualify the current partial assignment. Show how to solve the Satisfiability problem *efficiently*, if all clauses consist of at most two literals.

7.3. Problem (8) *Kolmogorov complexity*

Let U be a universal Turing machine. For a string $x \in \{0,1\}^*$ define the Kolmogorov complexity $K(x)$ of x by

$$K(x) = \min \{|p| \mid p \in \{0,1\}^*, U \text{ produces } x \text{ when started with } p\}.$$

So $K(x)$ is the length of the shortest program that produces x. We call x a *random string* iff $K(x) \geq |x|$ holds; i.e., x is "its shortest description".

a.) Show that random strings exist for all n.

b.) Determine the Kolmogorov complexity of the string $0^n 1^n$ in big–Oh notation.